Now consider string S_{next} among $S_1, \dots, S_k$ next to Steiner string S:

$$d_{\text{opt}}(S_{\text{next}}, S) = \min_{i=1,\dots,k} d_{\text{opt}}(S_i, S) \ .$$

Note that S_{next} exits, whereas we cannot feasibly compute its distance to S. This will thus not be the lower bound we are looking for. By choice of S_{next} we know that

$$\sum_{i=1}^{k} d_{\text{opt}}(S, S_i) \geq k d_{\text{opt}}(S, S_{\text{next}}) \ .$$

Former estimation for $a = \text{next}$ and the latter estimation together yield:

$$\begin{aligned}
\frac{\sum_{i=1}^{k} d_{\text{opt}}(S_i, S_{\text{next}})}{\sum_{i=1}^{k} d_{\text{opt}}(S_i, S)} &\leq \frac{\sum_{i=1}^{k} d_{\text{opt}}(S_i, S) - (k-2) d_{\text{opt}}(S, S_{\text{next}})}{\sum_{i=1}^{k} d_{\text{opt}}(S_i, S)} \\
&= 1 + \frac{(k-2) d_{\text{opt}}(S, S_{\text{next}})}{\sum_{i=1}^{k} d_{\text{opt}}(S_i, S)} \\
&\leq 1 + \frac{(k-2) d_{\text{opt}}(S, S_{\text{next}})}{k d_{\text{opt}}(S_{\text{next}}, S)} \\
&= 1 + \frac{k-2}{k} = \frac{2(k-1)}{k} \ .
\end{aligned}$$

Thus we have shown:

$$\sum_{i=1}^{k} d_{\text{opt}}(S_i, S_{\text{next}}) \leq \frac{2(k-1)}{k} \sum_{i=1}^{k} d_{\text{opt}}(S_i, S) \ .$$

□

6.3.3 2-Approximation Algorithm

A *center string* of $S_1, \dots, S_k$ is a string S_{center} taken from $S_1, \dots, S_k$ which minimizes summed alignment distances to all other strings.

$$\sum_{i=1}^{k} d_{\text{opt}}(S_{\text{center}}, S_i) = \min_{a=1,\dots,k} \sum_{i=1}^{k} d_{\text{opt}}(S_a, S_i)$$

Lemma 6.10.
The algorithm returning a center string S_{center} for $S_1, \dots, S_k$ is a 2-approximation algorithm.

Proof.